Response in kinetic Ising model to oscillating magnetic fields

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Ising models obeying Glauber dynamics in a temporally oscillating magnetic field are analyzed. In the context of stochastic resonance, the response in the magnetization is calculated by means of both a mean-field theory with linear-response approximation, and the time-dependent Ginzburg-Landau equation. Analytic results for the temperature and frequency dependent response, including the resonance temperature, compare favorably with simulation data.

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I. INTRODUCTION

Ising models with Glauber dynamics in an oscillating magnetic field were recently considered with Monte Carlo (MC) simulations in [1] and [2]. The phenomenon of stochastic resonance [3] was revealed, exhibiting a characteristic peak in the correlation function $C(T)$ between the external oscillating magnetic field and the magnetization $M(t)$ versus the temperature T of the system. The resonance temperature T_r (the temperature at which $C(T)$ has a maximum) was systematically computed as a function of the driving period, lattice size and driving amplitude, both for two-dimensional (2D) [1] and three-dimensional (3D) [2] systems. The one-dimensional (1D) case was analysed by Brey and Prados [4] in a linear-field approximation.

The present work is a natural continuation of those studies, considering analytically the 2D and 3D cases. We will present two approaches. The mean-field theory with linear response approximation will be discussed first. Then in 2D where the mean-field theory is not as good as in other dimensions, a more refined time-dependent Ginzburg-Landau (TDGL) approach will be presented, with significant improvements.

Recently, kinetic Ising systems in oscillating external fields have also been examined both experimentally and theoretically in [5]. The focus was on properties below the zero-field critical point, such as the frequency dependence of the probability distributions for the hysteresis-loop area and the residence time. The latter quantity for small systems in moderately weak fields suggests further evidences of stochastic resonance. Very recently, finite-size effects versus driving frequency are analyzed as a dynamical critical phenomena [6]. In contrast to these works, ours is focused on the temperature dependence above the zero-field critical point.

II. MEAN-FIELD THEORY AND LINEAR-RESPONSE APPROXIMATION

Our starting point is the master equation for the kinetic Ising model obeying Glauber dynamics [7]:

$$P(\sigma;t+1)-P(\sigma;t)= \sum_{\sigma'}\left[w(\sigma'\to\sigma)P(\sigma';t)-w(\sigma\to\sigma')P(\sigma;t)\right], \quad (1)$$

where $P(\sigma;t)$ is the joint probability of finding the spin configuration σ at time t, and w's are the transition rates between two configurations which differ by one spin flip. For the heat-bath algorithm, the rate function is chosen as

$$w(\sigma\to\sigma')=\frac{1}{1+e^{-\beta[E(\sigma)-E(\sigma')]}},$$

with $\beta=1/T$ (hereafter the Boltzmann constant $k\equiv1$), and $E(\sigma)$ is the energy of σ in a magnetic field h:

$$E(\sigma)=-J\sum_{\rm nn}S_iS_j-h(t)\sum_i S_i, \quad (2)$$

where $h(t)=A\sin(\omega t)$ and $\sum_{\rm nn}$ denotes a summation over nearest neighbors in a square or cubic lattice.

Let us denote the configuration σ by the values of the spins $S_1,S_2,...,S_V$, with system volume given by $V=N^d$. d is the spatial dimension of the system and N is its linear size. Since $S_i=\pm1$, it is easy to rewrite (1) as

$$\frac{d}{dt}P\left(S_1,S_2,.....S_V;t\right)=-\sum_{j=1}^{V}w_j(S_j)P(S_1,S_2,...,S_V;t) + \sum_{j=1}^{V}w_j(-S_j)P(S_1,S_2,...,-S_j,...,S_V;t) \quad (3)$$

with

$$w_j(S_j)=\frac{1}{2}[1-S_j\tanh(E_j/T)],$$

$$E_j=J\sum_{k=1}^{z}S_k+h, \quad (4)$$

where the last sum runs over the z nearest neighbors of the spin S_j, with $z = 2d$. Multiplying both sides of (3) by S_l and performing an ensemble average (denoted by $< \cdots >$), after some simple mathematical tricks, we get the basic equation for the Glauber dynamics:

$$\frac{d}{dt} < S_l >= - < S_l > + < \tanh(E_l/T) > . \tag{5}$$

Invoking the mean-field approximation, we replace E_l by $Jz < S > +h$ to get:

$$\frac{d}{dt} < S >= - < S > + \tanh[(h + T_c^{\rm MF} < S >)/T], \tag{6}$$

where $T_c^{\rm MF} = Jz$ is the mean-field critical temperature. In the absence of h, the magnetization is given by the stationary solution of the well-known equation:

$$< S >_0 = \tanh[T_c^{\rm MF} < S >_0 /T]. \tag{7}$$

For small $h(t)$, we may use the linear-response theory in (6) by first writing $< S > (t) =< S >_0 +\Delta S(t)$ and considering the $h/T \ll 1$ and $\Delta S/T \ll 1$ limits. Performing the Taylor expansion and keeping only the first-order terms, equation (6) becomes

$$\frac{d}{dt}\Delta S = -\frac{\Delta S}{\tau_{\rm MF}} + \frac{A}{T}(1- < S >_0^2)\sin(\omega t), \tag{8}$$

where

$$\tau_{\rm MF} = \frac{1}{1 - \frac{T_c^{\rm MF}}{T}(1- < S >_0^2)} \tag{9}$$

is the relaxation time. The solution can be found easily:

$$\Delta S(t) = \Delta S_0 \sin(\omega t - \theta_{\rm MF}), \tag{10}$$

with the phase shift and amplitude given by

$$\theta_{\rm MF} = \arctan(\omega \tau_{\rm MF}) \tag{11}$$

$$\Delta S_0 = \frac{A}{T}(1- < S >_0^2)\frac{1}{\sqrt{\frac{1}{\tau_{\rm MF}^2} + \omega^2}}. \tag{12}$$

The correlation function between the total magnetization $M = V < S >$ and the external field $h(t)$ can be computed:

$$\begin{aligned} C = \overline{M(t)h(t)} &\equiv (V\omega/2\pi)\int_0^{2\pi/\omega} \Delta S(t)h(t)dt \\ &= \frac{VA^2}{2T}(1- < S >_0^2)\frac{\tau_{\rm MF}}{1 + \omega^2\tau_{\rm MF}^2}. \end{aligned} \tag{13}$$

Here the overline denotes a temporal average over a period $P = 2\pi/\omega$. In the $T > T_c^{\rm MF}$ domain, $< S >_0 = 0$, thus C becomes:

$$C_{T>T_c^{\rm MF}} = \frac{VA^2}{2}\frac{T - T_c^{\rm MF}}{(T - T_c^{\rm MF})^2 + \omega^2 T^2}. \tag{14}$$

III. TIME-DEPENDENT GINZBURG-LANDAU APPROACH

Before comparing (13) to simulations, we present an alternative, continuum approach to compute C. For an Ising system with non-conservative order parameter (model A [8]), the time-dependent Ginzburg-Landau (TDGL) equation for the local magnetization density $\phi(\vec{r}, t)$ takes the following form:

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta \mathcal{H}}{\delta \phi} + \zeta, \tag{15}$$

$$\mathcal{H} = \int d\vec{r} \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}u\phi^2 + \frac{g}{4!}\phi^4 \right\}, \tag{16}$$

where $\mathcal{H}$ is the coarse grained Hamiltonian. For our present purpose, the white noise $\zeta(\vec{r}, t)$ which accounts for the effect of thermal fluctuations is irrelevant. Conventionally, parameters Γ, u and g in (16) are understood to be obtained by coarse graining the microscopic dynamics (1). For critical properties, the important temperature dependence in these parameters lies in $u \propto T - T_c^{\rm GL}$, giving rise to the spontaneous symmetry breaking below the critical temperature $T_c^{\rm GL}$. In order to compare with simulations, more precise dependences on T are required. To this end, we outline here a refined mean-field approach in the continuum limit. The same approach has been successfully applied to the two-species driven diffusive systems [9]. This approximation is expected to be good outside the critical region. However, this turns out to be not a serious handicap because the presence of an oscillating field prevents the system from building up critical correlations.

In a mean-field approximation, the joint probabilities in (1) are factorized into singlet probabilities $p(\vec{r}; t)$ for finding the spin up at site $\vec{r}$ at time t. This effectively produces the power series expansion of $\mathcal{H}$ in ϕ. $1-p$ gives the probability of finding the spin down. The continuum limit involves an expansion in the derivatives, such as:

$$p(x \pm 1, y; t) \to p(x, y; t) \pm \frac{\partial p(x, y; t)}{\partial x} + \frac{1}{2}\frac{\partial^2 p(x, y; t)}{\partial x^2} + \cdots$$

By identifying p as $(\phi+1)/2$, we obtain from (1) a kinetic equation for ϕ after some algebra. For $h = 0$, we find precisely the deterministic part of (15) with:

$$\Gamma = \frac{1}{8}(-2W_4 + 2W_{-4} - W_8 + W_{-8}), \tag{17}$$

$$u = \frac{1}{8\Gamma}(6W_0 + 12W_4 - 4W_{-4} + 5W_8 - 3W_{-8}), \tag{18}$$

$$g = \frac{3}{2\Gamma}(-6W_0 - 4W_4 + 4W_{-4} + 5W_8 + W_{-8}), \tag{19}$$

where $W_n \equiv 1/(1 + e^{n\beta J})$ contains the desired explicit T dependence. When a small uniform field h is applied, to $O(h)$ (neglecting a ϕ^5 term) we have finally the deterministic kinetic equation

$$\frac{\partial\phi}{\partial t} = -\Gamma\left\{-\nabla^2\phi + u\phi + \frac{g}{6}\phi^3 - \mu h\right\}, \qquad (20)$$

where $\mu = \beta(3W_0^2 + 4W_4W_{-4} + W_8W_{-8})/2\Gamma$. It is useful to note that Γ, g and μ in (20) are positive definite for all T, whereas u has one zero at $T_c^{\rm GL} \approx 3.0901J \approx 1.3618T_c$, where $T_c = -2/\ln(\sqrt{2}-1)J \approx 2.2692J$ is exact. This is an improvement over $T_c^{\rm MF} = 4J$ from the last section. Moreover, we reproduce the first few terms of the high-temperature series expansions of thermodynamic quantities such as the susceptibility and the relaxation time. In the $\beta \to 0$ limit, we recover the mean-field results of the last section: $u \approx 1/\beta J - 4$, $\Gamma \approx \beta J$, $g \approx 48(\beta J)^2$, and $\mu \approx 1/J$.

For small h and $T > T_c^{\rm GL}$, the nonlinear term $g\phi^3$ in (20) is negligible. The total magnetization $M(t) = \int d\vec{r}\,\phi(\vec{r},t) = \tilde{\phi}(\vec{q}=0,t)$ in response to an external field can then be computed easily, where $\tilde{\phi}$ denotes the spatial Fourier transform of ϕ. It satisfies $\partial M/\partial t = -\Gamma u M + \Gamma\mu\tilde{h}(\vec{q}=0,t)$. We readily find

$$M(t) = \frac{V\mu A\Gamma}{\sqrt{(\Gamma u)^2 + \omega^2}}\sin(\omega t - \theta_{\rm GL}), \qquad (21)$$

where the phase shift is $\theta_{\rm GL} = \arctan(\omega/\Gamma u)$. The correlation function with h is then given by

$$C_{T>T_c^{\rm GL}} = \frac{VA^2\Gamma^2\mu u}{2[(\Gamma u)^2 + \omega^2]}. \qquad (22)$$

Note that this coincides with the mean-field result (14) in the high-temperature limit.

For $T < T_c^{\rm GL}$, the term proportional to g is needed to break the symmetry, leading to the spontaneous magnetization $m = \sqrt{-6u/g}$. Linearizing about m, we find precisely the same form of C as $T > T_c^{\rm GL}$ except that u is replaced by $-2u$ in (22).

IV. DISCUSSION AND COMPARISON WITH SIMULATIONS

¿From the simulation data in [1] and [2], we learn that the system has a maximum response to external driving at a definite temperature T_r which depends on the driving frequency. Hence T_r can be designated as the *resonance temperature*. ¿From the analytically determined correlation functions in (14) and (22), we find two peaks in C at above and below the respective T_c, and also $C(T_c) = 0$, as shown in Fig. 1. This double-peak structure in C is consistent with simulations for larger lattice sizes (up to $N = 200$ for 2D and $N = 40$ for 3D) and with smaller steps in T than reported in [1] and [2]. The reason for missing the peak below T_c in our earlier simulations is due to the use of small lattice sizes. Note that the peak below T_c is much smaller than the one above and its position is less sensitive to the driving period. The reason for the overestimated theoretical values of the peaks below T_c may be accounted for by the frustration of the system to order in the presence of $h(t)$; such frustration steming from metastability has not been taken into account explicitly in our theories below T_c when we seek the symmetry-breaking solutions.

We believe that this also explains the discrepancy at T_c, where simulations show a small but finite $C(T)$. Finite-size effects are not of great concern here because, as mentioned above, the correlation length even at T_c is truncated by h. In simulations, we have checked the convergence in $C(T)$ for $N \geq 50$ in 2D.

Focusing on $T > T_c$ from now on, the TDGL predictions for $C(T)$ are more accurate than those of the mean-field theory in general. They both converge to the simulations in the tails at $T \gg T_c$ (see Fig. 1). In 3D the mean-field theory is already acceptable.

Turning our attention to the amplitude dependence, replotting the simulation data from [1] and [2] suggests that the height of the peak $C(T_r) \propto A^2$, in agreement with (14) and (22). For not too large frequencies and small A, the theoretical proportionality constant agrees well with simulations. For example, the slope of $C(T_r)/(VT_c)$ versus A^2/T_c^2 for $P = 50$ in 2D gives 0.92 from simulations [1], 0.96 from TDGL and 0.99 from mean-field approach. In 3D the same slope is 0.88 from simulations [2], and 1.29 from mean-field approach (In 3D the comparison are worse because T_r is much closer now to T_c.) This proportionality is a manifestation of the linear response of the system to h, which breaks down at large enough amplitudes. Our new simulations show that this happens for $A/T_c > 0.15$ in 2D for $P = 40$.

A quantity of significant interest is the resonance temperature $T_r(P)$. It can be determined analytically from (14)

$$T_r^{\rm MF} = T_c^{\rm MF}\left(1 + \sqrt{1 - \frac{1}{\omega^2+1}}\right), \qquad (23)$$

and numerically from (22) for $T_r^{\rm GL}$. These together with simulation results are presented in Fig. 2. The agreements are reasonable. As expected the mean-field approximation is quite good in 3D but in 2D the TDGL approximation is better.

The results in Fig. 2 confirm the earlier observation in [1] and [2] that for $P \to \infty$ we get $T_r \to T_c$. This result is also consistent with the one obtained by Brey and Prados [4] in 1D where the above limit becomes $T_r \to T_c = 0$. In the opposite limit $P \to 1$ (in unit of Monte Carlo steps $P \geq 1$) both the theory in 1D [4] and our approximations in 2D and 3D suggest $T_r \to$ const. Unfortunately, in [1] and [2] the wrong conclusion $T_r \to \infty$ was drawn in this limit. Similarly, the position of the peak below T_c also converges to T_c in the $P \to \infty$ limit.

In passing, we also derive [10] the relationship between the correlation function and the hysteresis-loop area $\mathcal{A}$:

$$\mathcal{A} = 2\pi C \mid \tan\theta \mid \qquad (24)$$

where θ is the phase shift between h and M. This then relates our results of C to that of $\mathcal{A}$ as observed in [5].

V. CONCLUSIONS

Using mean-field with linear-response and TDGL approximations, the characteristics of the resonance peaks observed in kinetic Ising models in oscillating magnetic fields [1,2] are reproduced. New simulations improve earlier results by confirming the analytically predicted double peaks. Focusing mostly on the behavior above T_c (where our approaches work better), we determine the dependence of the resonance temperature as a function of driving frequency and amplitude. We confirm the already predicted result in [1,2] that $T_r \to T_c$ for the limit of practically interesting driving frequencies ($P \to \infty$), and corrected the wrong extrapolation in the opposite limit $P \to 1$. We introduce a refined TDGL approach which improves significantly the mean-field results in 2D, but in 3D the mean-field approximation is already acceptable. We have thus demonstrated that the stochastic resonance in kinetic Ising models above T_c can be understood by means of rather simple theoretical approaches for small driving amplitudes.

VI. ACKNOWLEDGMENTS

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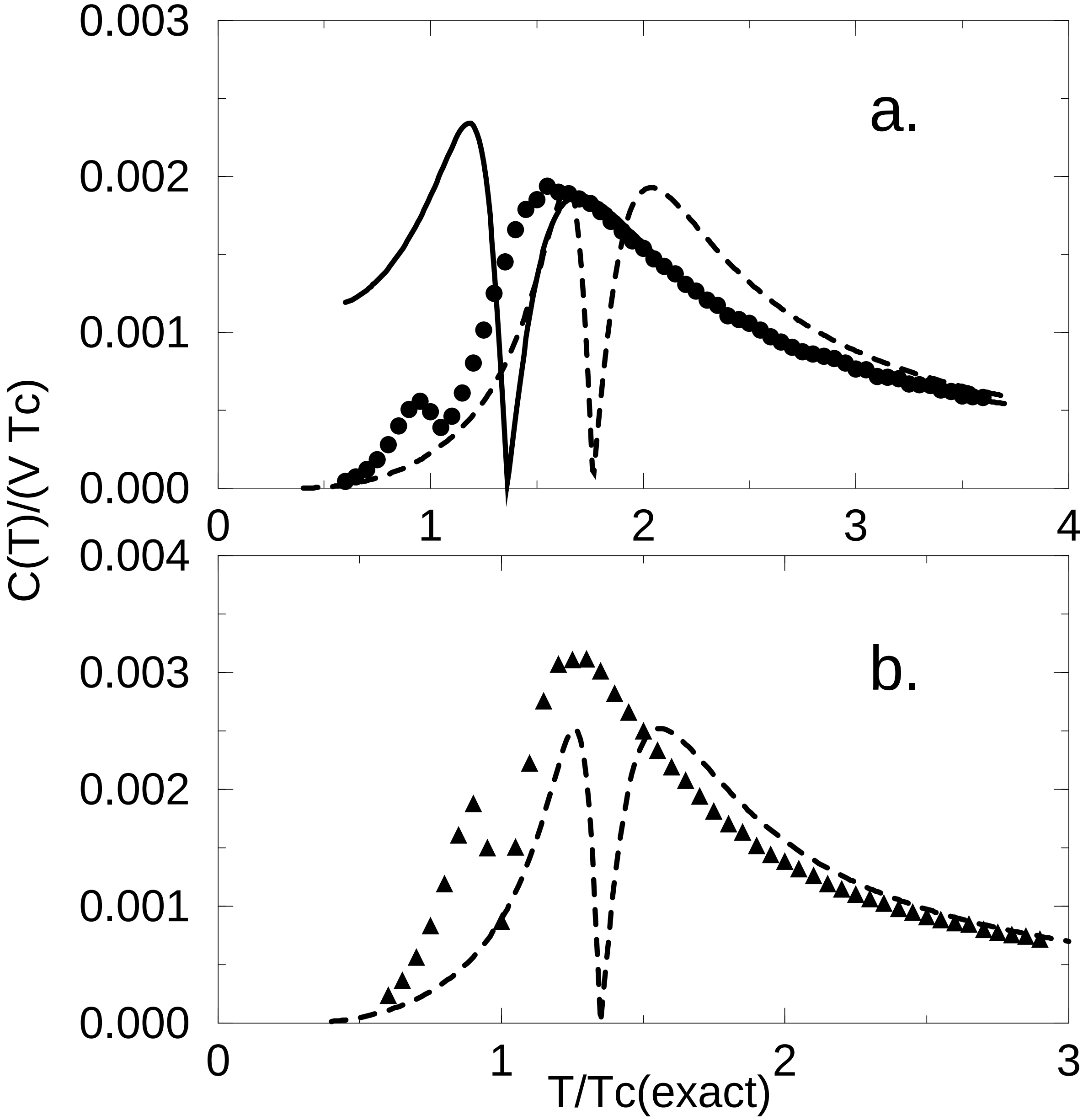

FIG. 1. $C(T)/(VT_c)$ versus temperature for $P = 40$ and $A = 0.05T_c$ for 2D in (a) and 3D in (b). Dots are MC simulation results in 2D ($N = 200$), triangles are MC simulations in 3D ($N = 40$), continuous line is from TDGL approximation and the dashed line is the mean-field result. The higher peak for TDGL than mean-field theory below T_c is due to our dropping the ϕ^5 term in (20).

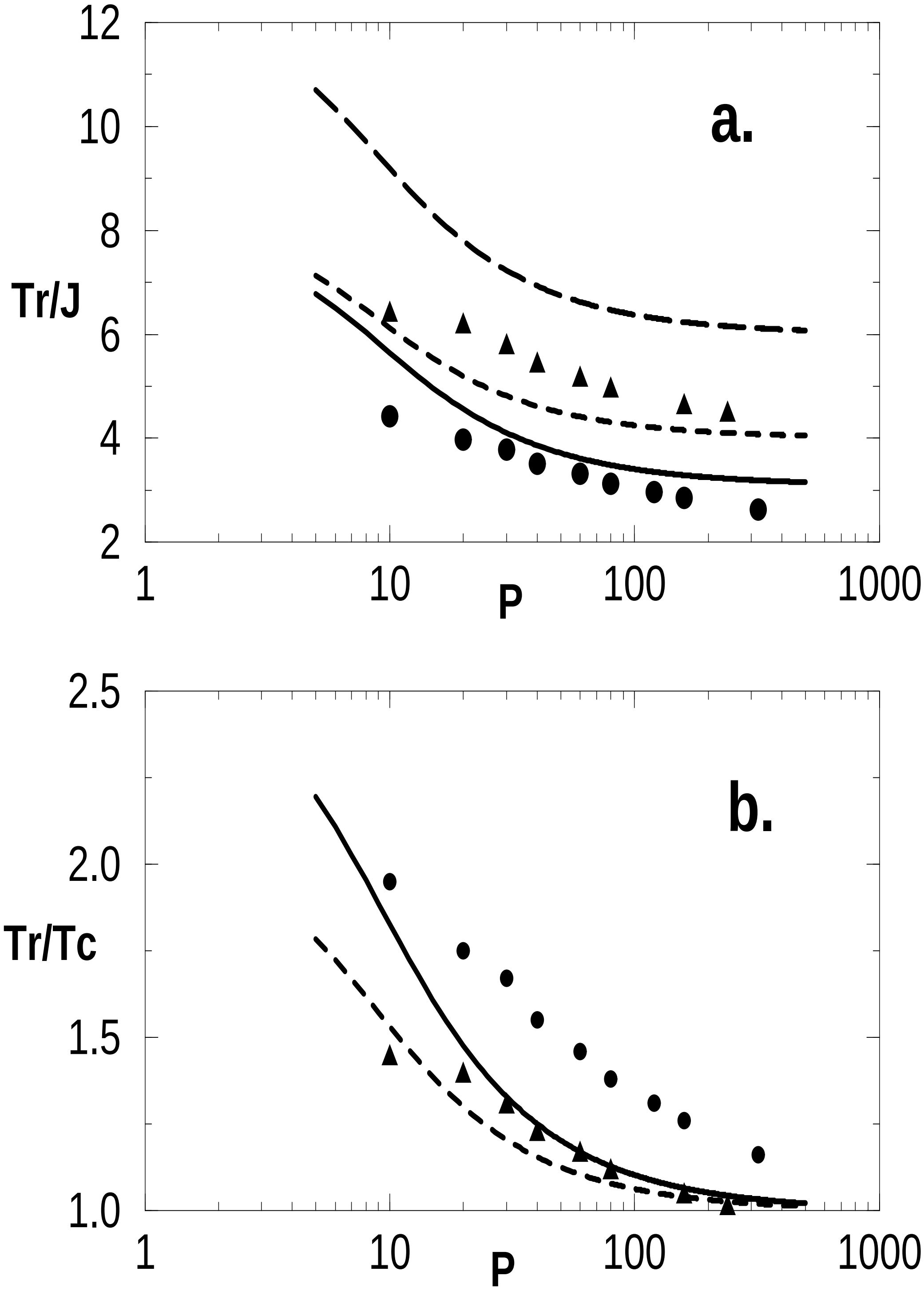

FIG. 2. Resonance temperature above T_c versus driving period P for $A = 0.05$, on absolute scale T_r/J in (a) and on relative scale T_r/T_c in (b). The long-dashed and short-dashed lines in (a) are the mean-field results for 3D and 2D respectively, in the rest the symbols mean the same as in Fig. 1.